

UNIMODULARITY OF ZEROS OF SELF-INVERSE POLYNOMIALS

MATILDE N. LALÍN AND CHRIS J. SMYTH

ABSTRACT. We generalise a necessary and sufficient condition given by Cohn for all the zeros of a self-inversive polynomial to be on the unit circle. Our theorem implies some sufficient conditions found by Lakatos, Losonczi and Schinzel. We apply our result to the study of a polynomial family closely related to Ramanujan polynomials, recently introduced by Gun, Murty and Rath, and studied by Murty, Smyth and Wang and Lalín and Rogers. We prove that all polynomials in this family have their zeros on the unit circle, a result conjectured by Lalín and Rogers on computational evidence.

1. INTRODUCTION

A *self-inversive* polynomial of degree d is a nonzero complex polynomial $P(z)$ that satisfies

$$P(z) = \varepsilon z^d \overline{P}(1/z) \quad (1)$$

for some constant ε . So if $P(z) = \sum_{j=0}^d A_j z^j$ is self-inversive, then $A_j = \varepsilon \overline{A_{d-j}}$ for $j = 0, \dots, d$. In particular $A_0 = \varepsilon \overline{A_d}$ and $\overline{A_d} = \bar{\varepsilon} A_0$, so that ε is necessarily of modulus 1. It is easy to check that if a polynomial has all its zeros on the unit circle then it is self-inversive. In fact, Cohn [5] proved that a polynomial $P(z)$ has all its zeros on the unit circle if and only if it is self-inversive and its derivative $P'(z)$ has all its zeros in the closed unit disk $|z| \leq 1$. One might think that Cohn's result completely settles the matter. Indeed, Cohn's condition on P' can be verified for a specific self-inversive polynomial, for instance by finding the zeros of P' , or checking that the Mahler measure of P' is equal to the modulus of its leading coefficient. However, it may not be possible to use these methods for whole parametrized families of polynomials. We prove here an extension of Cohn's result – see Theorem 1 below – which turns out to be more flexible than Cohn's Theorem for applications. We apply it to some polynomial families,

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including the polynomial family P_k given by

$$P_k(z) = \frac{(2\pi)^{2k-1}}{(2k)!} \sum_{j=0}^k (-1)^j B_{2j} B_{2k-2j} \binom{2k}{2j} z^{2j} + \zeta(2k-1) (z^{2k-1} + (-1)^k z). \quad (2)$$

Here, as usual, the B_{2j} are Bernoulli numbers. It was known from work of Lalín and Rogers [11] that the polynomials in this family had all their zeros on the unit circle for $k \leq 1000$. They conjectured that this was true for all k . However, this conjecture had resisted previous attempts to prove it. We do this in Theorem 8 below, as an application of our main theorem (Theorem 1).

There have also been a number of results in recent years that provide sufficient, easier-to-verify conditions for a self-inversive polynomial to have its zeros on the unit circle. Lakatos [7] proved that a reciprocal polynomial $\sum_{j=0}^d A_j z^j$ with real coefficients that satisfies

$$|A_d| \geq \sum_{j=0}^d |A_j - A_d|, \quad (3)$$

has its zeros on the unit circle. Moreover, if the inequality is strict, the zeros are simple. Schinzel [18] improved Lakatos' result. His result – see Corollary 5 and Remark 6 below – follows from Theorem 4.

On the other hand, Lakatos and Losonczy [9] proved that a reciprocal polynomial satisfying

$$|(1 + \alpha)A_d| \geq \sum_{j=1}^{d-1} |A_j - (1 - \alpha)A_d| \quad (4)$$

for some $0 \leq \alpha \leq 1$, has all its zeros on the unit circle. We have not been able to deduce their result using our approach, except in the case $\alpha = 0$, see (3) and Remark 6 below, and in the case $\alpha = 1$, which is Corollary 3. Their result was combined with Schinzel's into a more general statement, proved in [10].

We will prove another result (Theorem 4) of the same general type that extends Schinzel's criterion in a different direction.

For other results concerning self-inversive and reciprocal polynomials, see the books by Marden [12, pp. 201–206], Rahman and Schmeisser [15] and Schinzel [17], as well as the papers by Ancochea [1], Bonsall and Marden [3, 4] and O'Hara and Rodriguez [14].

In Section 2 we state and prove our main result (Theorem 1), and deduce various consequences of it, including Theorem 4 and Lemma 7. In Section 3 we apply Lemma 7 to prove that the polynomial family P_k given by (2) have

all their zeros on the unit circle. Then in Section 4 we give some details of how the same result, due originally to Lalín and Rogers [11], can be proved in a similar way for two other polynomial families Q_k and W_k .

2. RESULTS

Our main theorem is the following.

Theorem 1. *Let $h(z)$ be a nonzero complex polynomial of degree n having all its zeros in the closed unit disk $|z| \leq 1$. Then for $d > n$ and any λ on the unit circle, the self-inversive polynomial*

$$P^{\{\lambda\}}(z) = z^{d-n}h(z) + \lambda h^*(z) \quad (5)$$

has all its zeros on the unit circle.

Conversely, given a self-inversive polynomial $P(z)$ having all its zeros on the unit circle, there is a polynomial h having all its zeros in $|z| \leq 1$ such that P has a representation (5). In particular, we can take $h(z) = \frac{1}{d}P'(z)$.

Here $h^*(z) = z^n \overline{h(1/z)}$.

Proof. Assume first that the polynomial h , of degree n , has all its zeros in $|z| < 1$. Then $z^{d-n}h(z)$ has all of its zeros in the open unit disk while $h^*(z)$ has all of its zeros with absolute value greater than 1. Now take z such that $|z| = 1$. We have

$$|h^*(z)| = |\overline{h(1/z)}| = |\overline{h(z)}| = |z^{d-n}h(z)|.$$

Assume for the time being that λ has absolute value greater (respectively less) than 1. Then $|z^{d-n}h(z)|$ is less (respectively greater) than $|\lambda h^*(z)|$. Hence, by Rouché's Theorem, $P^{\{\lambda\}}(z)$ has all of its zeros in $|z| > 1$ (respectively all of its zeros in $|z| < 1$). As the zeros of $P^{\{\lambda\}}$ are continuous functions of λ , we see that when $|\lambda| = 1$ then $P^{\{\lambda\}}$ must have all its zeros on the unit circle.

The result under the weaker assumption that h has all its zeros in the closed unit disc $|z| \leq 1$ then follows by continuity.

Conversely, given P self-inversive of degree d with all its zeros on the unit circle, we note that differentiating (1) gives

$$P(z) = \frac{z}{d}P'(z) + \varepsilon \frac{z^{d-1}}{d} \overline{P'}(1/z), \quad (6)$$

which is of the form (5) with $h(z) = \frac{1}{d}P'(z)$, $\lambda = \varepsilon$ and $n = d - 1$. Further, the zeros of P' certainly all lie in $|z| \leq 1$. This is because the zeros of P' lie within the convex hull of the zeros of P , a result due originally to Gauss and Lucas — see [12, pp. 23–24] and [15, pp. 72–73, pp. 92–93] for relevant

references. For completeness, and because of its elegance, we now reproduce a proof, by Cesàro, of this latter result, taken from [15].

Let $P(z)$ have zeros $\alpha_1, \dots, \alpha_d$, and suppose that $P'(\beta) = 0$. If β equals some α_j then β is clearly in the convex hull of all the α_i . So we can assume that $P(\beta) \neq 0$, and then on logarithmic differentiation we have $P'(z)/P(z) = \sum_i \frac{1}{z-\alpha_i}$, and hence, putting $z = \beta$ and taking complex conjugates, that $\sum_i \frac{1}{\overline{\beta-\alpha_i}} = 0$. Then $\sum_i \frac{\beta-\alpha_i}{|\beta-\alpha_i|^2} = 0$, giving $\beta = \sum_i \lambda_i \alpha_i$, where

$$\lambda_i = \frac{|\beta - \alpha_i|^{-2}}{\sum_j |\beta - \alpha_j|^{-2}}.$$

Thus the λ_i are all positive and sum to 1. \square

It is not the case in general that if (5) holds for a particular h , and P has all its zeros on the unit circle then h must have all its zeros in $|z| \leq 1$. For example, it is known (see e.g., [16, p. 9]) that the polynomial $P(z) = z^k(z^3 - z - 1) + (z^3 + z^2 - 1)$ has all its zeros on the unit circle for $k = 0, 1, \dots, 7$, while $h(z) = z^3 - z - 1$ has a zero $1.3247\dots > 1$. However, in that direction we can say the following.

Observation 2. For any $d > n$, let

$$P_d^{\{\lambda\}}(z) = z^{d-n}h(z) + \lambda h^*(z).$$

If there is a $K > 0$ such that for every $d > K$, $P_d^{\{\lambda\}}(z)$ has all its zeros on the unit circle, then $h(z)$ has all its zeros in the unit circle $|z| \leq 1$.

Proof. Assume that $h(z)$ has a zero z_0 with $|z_0| > 1$. Take $\delta < |z_0| - 1$, so that $|z_0| - \delta > 1$. Then for z on the circle $|z - z_0| = \delta$ and d sufficiently large we have

$$|z^{d-n}h(z)| \geq (|z_0| - \delta)^{d-n}|h(z)| > |\lambda h^*(z)|.$$

Hence, by Rouché's Theorem, $P^{(\lambda)}(z)$ has the same number of zeros in the disc $|z - z_0| < \delta$ as $z^{d-n}h(z)$ has, namely at least one. This disc is completely outside the unit circle. \square

As a simple consequence of Theorem 1, we obtain the following known result.

Corollary 3 (Lakatos and Losonczi [8]). *A self-inversive polynomial $P(z) = \sum_{j=0}^d A_j z^j$ satisfying*

$$|A_d| \geq \frac{1}{2} \sum_{j=1}^{d-1} |A_j| \tag{7}$$

has all its zeros on the unit circle.

Proof. We take

$$h(z) = \begin{cases} A_d z^{\frac{d}{2}} + A_{d-1} z^{\frac{d}{2}-1} + \dots + A_{\frac{d}{2}+1} z + \frac{1}{2} A_{\frac{d}{2}}, & d \text{ even,} \\ A_d z^{\frac{d-1}{2}} + A_{d-1} z^{\frac{d-3}{2}} + \dots + A_{\frac{d+1}{2}}, & d \text{ odd.} \end{cases}$$

Then, since by (7) the leading coefficient of $h(z)$ is at least as big as the sum of the moduli of the other coefficients, $h(z)$ has no zeros in $|z| > 1$. Now $P(z)/A_d = z^{\lfloor \frac{d+1}{2} \rfloor} h(z) + \varepsilon h^*(z)$, where $\varepsilon = A_0/\overline{A_d}$. Therefore, by Theorem 1, $P(z)$ has all its zeros on the unit circle. \square

We can also deduce the next result from Theorem 1.

Theorem 4. *A self-inversive polynomial $P(z) = \sum_{j=0}^d A_j z^j$ satisfying*

$$|A_d| \geq \frac{1}{2} \inf_{\substack{\mu \in \mathbb{C} \\ |\mu|=1}} \sum_{j=0}^{d-1} |A_j - \mu A_{j+1}| \quad (8)$$

has all its zeros on the unit circle.

Proof. We first consider the case $\mu = 1$, and look at $P(z)(z-1)$. We take

$$h(z) = \begin{cases} A_d z^{\frac{d}{2}} + (A_{d-1} - A_d) z^{\frac{d}{2}-1} + \dots \\ + \left(A_{\frac{d}{2}} - A_{\frac{d}{2}+1} \right), & d \text{ even,} \\ A_d z^{\frac{d+1}{2}} + (A_{d-1} - A_d) z^{\frac{d-1}{2}} + \dots \\ + \left(A_{\frac{d+1}{2}} - A_{\frac{d+3}{2}} \right) z + \frac{1}{2} \left(A_{\frac{d-1}{2}} - A_{\frac{d+1}{2}} \right), & d \text{ odd.} \end{cases}$$

Again, the absolute value of the leading coefficient of $h(z)$ is at least as big as the sum of the moduli of the other coefficients. This implies that $h(z)$ has no zeros in $|z| > 1$ and we can apply the Theorem 1 with $\lambda = -\varepsilon$ to conclude that $P(z)(z-1)$, and therefore $P(z)$, has all its zeros on the unit circle.

To obtain the result in general, let $Q(z) := P(\mu z)$ and apply what has been proved to $Q(z)$, using the fact that $|\mu^j A_j - \mu^{j+1} A_{j+1}| = |A_j - \mu A_{j+1}|$. \square

Corollary 5 (Schinzel [18]). *A self-inversive polynomial $P(z) = \sum_{j=0}^d A_j z^j$ satisfying*

$$|A_d| \geq \inf_{\substack{c, \mu \in \mathbb{C} \\ |\mu|=1}} \sum_{j=0}^d |c A_j - \mu^{d-j} A_d|, \quad (9)$$

must have all of its zeros on the unit circle.

Proof. Consider the self-inversive polynomial $P(z) = \sum_{j=0}^d A_j z^j$ and assume that (9) is satisfied. Then we claim that (8) holds. We first check this for $\mu = 1$. Indeed, by applying twice the triangle inequality,

$$\begin{aligned} \sum_{j=0}^{d-1} |A_j - A_{j+1}| &\leq \sum_{j=0}^{d-1} |A_j - 1/c A_d| + \sum_{j=0}^{d-1} |1/c A_d - A_{j+1}| \\ &= 2 \sum_{j=0}^d |A_j - 1/c A_d| - 2|1 - 1/c||A_d| \\ &\leq 2|1/c A_d| - 2|1 - 1/c||A_d| \\ &\leq 2|A_d|. \end{aligned}$$

Then, again applying the result to $P(\mu z)$ for general μ on the unit circle gives the full result. \square

Remark 6. The condition (3) of Lakatos is the special case $c = \mu = 1$ of (9).

We next show that the result of Theorem 1 still holds if $P^{\{\lambda\}}(z)$ is perturbed by a small self-inversive ‘error’ polynomial.

Lemma 7. *Let h and λ be as in Theorem 1, with $|h(z)| \geq c > 0$ for $|z| = 1$. Let $e(z)$ be a polynomial of degree m such that $|e(z)| \leq c$ for $|z| = 1$. Then for $k > \max\{m, n\}$, the self-inversive polynomial*

$$z^{2k-n}h(z) + z^k e(z) + \lambda(h^*(z) + z^{k-m}e^*(z))$$

has all its zeros on the unit circle.

Proof. We first assume that for some positive $c' < c$ we have $|e(z)| \leq c' < c$ for all z with $|z| = 1$. Now $h_e(z) = z^{k-n}h(z) + e(z)$ is a polynomial of degree k . Because $|z^{k-n}h(z)| \geq c > c' \geq |e(z)|$ for $|z| = 1$, Rouché’s Theorem tells us that h_e has all its zeros in the open unit disk $|z| < 1$. Also $h_e^*(z) = z^k h_e(1/z) = h^*(z) + z^{k-m}e^*(z)$. Now apply Theorem 1 with h replaced by h_e and d replaced by k .

The general case, where we assume only that $|e(z)| \leq c$ for $|z| = 1$, then follows by continuity. \square

3. APPLICATION TO THE POLYNOMIALS P_k

Let $k \geq 2$, and, as in [11], define $P_k(z)$ by (2). The study of this polynomial is motivated by the fact that it appears in a formula by Ramanujan

[2, p. 276]:

$$\frac{1}{2z^k} P_k(z) = (-z)^{-(k-1)} \sum_{n=1}^{\infty} \frac{1}{n^{2k-1}(e^{2\pi n z} - 1)} - z^{k-1} \sum_{n=1}^{\infty} \frac{1}{n^{2k-1}(e^{2\pi n/z} - 1)}, \quad (10)$$

valid for $z \notin i\mathbb{Q}$. A variant of the polynomial $P_k(z)$ (without the term with the ζ -value) was first considered by Gun, Murty, and Rath [6] in the context of expressing the special value of the ζ -function as an Eichler integral that could yield information about the algebraic nature of the number. Murty, Smyth, and Wang [13] studied this variant of the polynomial and found that all but four of its zeros lie on the unit circle. Finally, other variants of $P_k(z)$ were considered in [11], and were shown to have all their zeros on the unit circle. However, the methods from [11] were not sufficient to prove that $P_k(z)$ itself has all its zeros on the unit circle.

Theorem 8. *For all $k \in \mathbb{N}$, the polynomial P_k has all its zeros on the unit circle.*

We need the following straightforward bounds. Put $q_j = \frac{\zeta(2j)\zeta(2k-2j)}{\zeta(2k)}$ for $j = 0, 1, \dots, k$ and $\delta_j = q_j - \zeta(2j)$. Note that $\delta_0 = 0$ and $\delta_j > 0$ for $0 < j < k$.

Lemma 9. (i) *For $n \geq 2$ we have*

$$1 < \zeta(n) < 1 + \frac{n+1}{n-1} \cdot 2^{-n}.$$

(ii) *For $k \geq 2$ and $j = 1, 2, \dots, k-1$ we have*

$$0 < \frac{\zeta(2k-2j)}{\zeta(2k)} - 1 < 3 \cdot 4^{j-k}.$$

(iii) *For $k \geq 11$ we have*

$$0 < \frac{\zeta(2k-1)}{\zeta(2k)} - 1 < \frac{11}{5} \cdot 4^{-k}.$$

(iv) *For $k \geq 4$ and $j = 1, 2, \dots, k-1$ we have*

$$|\delta_{j-1} - \delta_j| < \begin{cases} 21 \cdot 4^{-k} & \text{if } j = 1; \\ 3 \cdot 4^{-k} \left(4^j + \frac{2j-1}{2j-3} \right) & \text{if } j \geq 2. \end{cases}$$

(v) *For $k \geq 2$ and $2 \leq j \leq k/2$ we have $q_j = q_{k-j}$ and*

$$|q_{j-1} - q_j| < 3 \cdot 4^{-k} \left(4^j + \frac{2j-1}{2j-3} \right) + \frac{2j-1}{2j-3} \cdot 4^{1-j}.$$

(vi) *For $k \geq 4$ and $4 \leq r \leq k$ we have*

$$\sum_{j=1}^r |\delta_{j-1} - \delta_j| < 5 \cdot 4^{r-k}.$$

(vii) For $k \geq 10$ and $r \geq 4$ we have

$$\sum_{j=r+1}^{\lfloor \frac{k}{2} \rfloor} |q_{j-1} - q_j| < 5 \cdot 2^{-k} + \frac{12}{7} \cdot 4^{-r}.$$

Proof. Part (i)-(iii) are easy – see [13, Lemmas 4.4 and 4.6] for (i) and (ii). For (iv), we have, using (i) and (ii), that

$$\begin{aligned} \delta_j &= \zeta(2j) \left(\frac{\zeta(2k-2j)}{\zeta(2k)} - 1 \right) \\ &< \left(1 + \frac{2j+1}{2j-1} \cdot 4^{-j} \right) \cdot 3 \cdot 4^{j-k} \\ &= 3 \cdot 4^{-k} \left(4^j + \frac{2j+1}{2j-1} \right). \end{aligned}$$

Hence

$$|\delta_0 - \delta_1| = \delta_1 < 21 \cdot 4^{-k},$$

while in general

$$|\delta_{j-1} - \delta_j| \leq \max(\delta_{j-1}, \delta_j),$$

from which the result for $j \geq 2$ follows.

For (v), we have

$$|q_{j-1} - q_j| \leq |\delta_{j-1} - \delta_j| + \zeta(2j-2) - \zeta(2j),$$

which gives the result using (ii) and (iv).

For (vi), we have, using (iv), that

$$\begin{aligned} \sum_{j=1}^r |\delta_{j-1} - \delta_j| &< 21 \cdot 4^{-k} + 3 \cdot 4^{-k} \sum_{j=2}^r \left(4^j + \frac{2j-1}{2j-3} \right) \\ &< 4^{r-k} \left(21 \cdot 4^{-r} + 3 \sum_{j=-\infty}^r 4^{j-r} + 3 \cdot 3 \cdot (r-1) \cdot 4^{-r} \right) \\ &< 4^{r-k} (21/4^4 + 4 + 27/4^4) \\ &< 5 \cdot 4^{r-k}. \end{aligned}$$

For (vii), as we have $j \geq 5$ in the summand, and $r \geq 4$, we obtain, using (v), that

$$\begin{aligned} \sum_{j=r+1}^{\lfloor \frac{k}{2} \rfloor} |q_{j-1} - q_j| &\leq \sum_{j=r+1}^{\lfloor \frac{k}{2} \rfloor} \left(3 \cdot 4^{-k} \left(4^j + \frac{2j-1}{2j-3} \right) + \frac{2j-1}{2j-3} \cdot 4^{1-j} \right) \\ &< \sum_{j=-\infty}^{\lfloor \frac{k}{2} \rfloor} 3 \cdot 4^{j-k} + 3 \cdot 4^{-k} \cdot \frac{9}{7} \left(\left\lfloor \frac{k}{2} \right\rfloor - r \right) + \frac{9}{7} \sum_{j=r+1}^{\infty} 4^{1-j} \\ &\leq 3 \cdot 4^{-k/2} \cdot \frac{4}{3} + 2^{-k} \cdot \left(\frac{27}{7} \left(\frac{k}{2} - 4 \right) 2^{-k} \right) + \frac{9}{7} \cdot 4^{-r} \cdot \frac{4}{3} \\ &\leq 2^{-k}(4+1) + \frac{12}{7} \cdot 4^{-r}. \end{aligned}$$

□

We also need the standard identity

$$\frac{B_{2j}}{(2j)!} = (-1)^{j+1} \frac{2\zeta(2j)}{(2\pi)^{2j}},$$

valid for all $j \geq 0$, since $B_0 = 1$ and $\zeta(0) = -1/2$. From this we see that

$$P_k(z) = (-1)^k \frac{2}{\pi} \sum_{j=0}^k (-z^2)^j \zeta(2j) \zeta(2k-2j) + \zeta(2k-1) (z^{2k-1} + (-1)^k z).$$

Thus P_k has leading coefficient $-\zeta(2k)/\pi$. To show that P_k has all its zeros on the unit circle it is sufficient to show that the monic polynomial $M_k(z) = -\frac{\pi}{\zeta(2k)}(z^2 + 1)P_k(z)$ has all its zeros on the unit circle. (Most of the coefficients of M_k are very small, making it easier to work with than $-\frac{\pi}{\zeta(2k)}P_k$, whose coefficient of z^{2j} is close to $2(-1)^{j+k+1}$ for most j .) We easily calculate that

$$\begin{aligned} M_k(z) &= z^{2k+2} + (-1)^k - \frac{\pi\zeta(2k-1)}{\zeta(2k)} (z^{2k+1} + z^{2k-1} + (-1)^k z^3 + (-1)^k z) \\ &\quad + 2 \sum_{j=1}^k (-1)^j z^{2k+2-2j} (q_{j-1} - q_j). \end{aligned}$$

Hence, using the fact that for k odd and $j = (k+1)/2$ we have $q_{j-1} - q_j = 0$, we obtain

$$\begin{aligned} z^{-(k+1)} M_k(z) &= z^{k+1} + (-1)^k z^{-(k+1)} - \frac{\pi\zeta(2k-1)}{\zeta(2k)} (z^k + z^{k-2} + (-1)^k (z^{2-k} + z^{-k})) \\ &\quad + 2 \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (q_{j-1} - q_j) (-1)^j (z^{k+1-2j} + (-1)^k z^{-(k+1-2j)}) \\ &= z^{k+1} + (-1)^k z^{-(k+1)} - \pi (z^k + z^{k-2} + (-1)^k (z^{2-k} + z^{-k})) \\ &\quad + 2 \sum_{j=1}^r (\zeta(2j-2) - \zeta(2j)) (-1)^j (z^{k+1-2j} + (-1)^k z^{-(k+1-2j)}) \\ &\quad + 2 \sum_{j=r+1}^{\lfloor \frac{k}{2} \rfloor} (q_{j-1} - q_j) (-1)^j (z^{k+1-2j} + (-1)^k z^{-(k+1-2j)}) \\ &\quad + 2 \sum_{j=1}^r (\delta_{j-1} - \delta_j) (-1)^j (z^{k+1-2j} + (-1)^k z^{-(k+1-2j)}) \\ &\quad - \pi \left(\frac{\zeta(2k-1)}{\zeta(2k)} - 1 \right) (z^k + z^{k-2} + (-1)^k (z^{2-k} + z^{-k})) \\ &= z^{-(k+1)} H_r(z) + z^{-(k+1)} E_r(z), \end{aligned}$$

where

$$\begin{aligned}
H_r(z) &= z^{2k+2} + (-1)^k - \pi(z^{2k+1} + z^{2k-1} + (-1)^k(z^3 + z)) \\
&\quad + 2 \sum_{j=1}^r (\zeta(2j-2) - \zeta(2j))(-1)^j(z^{2k+2-2j} + (-1)^k z^{2j}) \\
E_r(z) &= -\pi \left(\frac{\zeta(2k-1)}{\zeta(2k)} - 1 \right) (z^{2k+1} + z^{2k-1} + (-1)^k(z^3 + z)) \\
&\quad + 2 \sum_{j=1}^r (\delta_{j-1} - \delta_j)(-1)^j(z^{2k+2-2j} + (-1)^k z^{2j}) \\
&\quad + 2 \sum_{j=r+1}^{\lfloor \frac{k}{2} \rfloor} (q_{j-1} - q_j)(-1)^j(z^{2k+2-2j} + (-1)^k z^{2j}).
\end{aligned}$$

Here H_r is the main polynomial, with E_r the error polynomial.

We can rewrite $H_r(z)$ as

$$H_r(z) = z^{2k+2-2r} h_r(z) + (-1)^k h_r^*(z),$$

where

$$h_r(z) = z^{2r} - \pi z^{2r-1} - \pi z^{2r-3} + 2 \sum_{j=1}^r (\zeta(2j-2) - \zeta(2j))(-1)^j z^{2r-2j}, \quad (11)$$

and $E_r(z)$ as

$$E_r(z) = z^{k+2} e_r(z) + (-1)^k z e_r^*(z),$$

where

$$\begin{aligned}
e_r(z) &= -\pi \left(\frac{\zeta(2k-1)}{\zeta(2k)} - 1 \right) (z^{k-1} + z^{k-3}) + 2 \sum_{j=1}^r (\delta_{j-1} - \delta_j)(-1)^j z^{k-2j} \\
&\quad + 2 \sum_{j=r+1}^{\lfloor \frac{k}{2} \rfloor} (q_{j-1} - q_j)(-1)^j z^{k-2j}.
\end{aligned}$$

We now take $r = 4$. We then have the following bound.

Lemma 10. *For $k \geq 11$ and $|z| = 1$ we have $|e_4(z)| \leq 0.019$.*

Proof. Take z with $|z| = 1$. Applying Lemma 9 (iii), (vi) and (vii) we have

$$\begin{aligned}
|e_r(z)| &\leq \left| -\pi \left(\frac{\zeta(2k-1)}{\zeta(2k)} - 1 \right) (z^{k-1} + z^{k-3}) \right| + \left| 2 \sum_{j=1}^r (\delta_{j-1} - \delta_j)(-1)^j z^{k-2j} \right| \\
&\quad + \left| 2 \sum_{j=r+1}^{\lfloor \frac{k}{2} \rfloor} (q_{j-1} - q_j)(-1)^j z^{k-2j} \right| \\
&< 2\pi \cdot \frac{11}{5} \cdot 4^{-k} + 2 \cdot 5 \cdot 4^{r-k} + 2 \cdot 5 \cdot 2^{-k} + 2 \cdot \frac{12}{7} \cdot 4^{-r},
\end{aligned}$$

which is less than 0.019 for $r = 4$ and $k \geq 11$. \square

Proof of Theorem 8. The result is known to be true for $k \leq 10$ – see [11]. The polynomial

$$\begin{aligned} h_4(z) = & z^8 - \pi z^7 + \left(1 + \frac{\pi^2}{3}\right) z^6 - \pi z^5 + \left(\frac{\pi^2}{3} - \frac{\pi^4}{45}\right) z^4 \\ & + \left(\frac{2\pi^6}{945} - \frac{\pi^4}{45}\right) z^2 + \left(\frac{2\pi^6}{945} - \frac{\pi^8}{4725}\right) \end{aligned}$$

given by (11) has all its zeros of modulus less than 1. Furthermore, it is a matter of routine calculation to find that its minimum on the unit circle occurs at $z_0, \overline{z_0} \approx e^{\pm 0.20325951i}$, where $|h_4(z_0)| = 0.0214\dots > 0.020$.

So for $k \geq 11$ we can apply Lemma 7 with $h = h_4$, $e = e_4$ and $c = 0.020$ to obtain the required result. \square

4. APPLICATIONS TO OTHER POLYNOMIALS

The method described in this work can be also used to study the other families of polynomials that appear in the statement of Claim 1.1 of [11]. First notice that the coefficients can be thought of as special values of variations of the Riemann zeta function or a Dirichlet L -function.

$$\begin{aligned} (-1)^k \frac{\pi}{2^{2k+1}} Q_k(z) := & \sum_{j=1}^{k-1} \eta_0(2j) \eta_0(2k-2j) (-z^2)^j \\ & + (-1)^k \frac{\pi}{4} \eta_0(2k-1) (z^{2k-1} + (-1)^k z); \end{aligned} \quad (12)$$

$$\frac{(-1)^k}{4} Y_k(z) = \sum_{j=1}^{k-1} \eta_0(2j) \eta_0(2k-2j) z^j; \quad (13)$$

$$(-1)^k \frac{\pi}{2^{2k+1}} W_k(z) = \sum_{j=0}^k \eta(2j) \eta(2k-2j) (-z^2)^j; \quad (14)$$

$$\frac{(-1)^k}{(2k)!4} \left(\frac{\pi}{2}\right)^{2k+2} S_k(z) = \sum_{j=0}^k L(2j+1, \chi_4) L(2k-2j+1, \chi_4) z^j. \quad (15)$$

We have used that

$$L(2j+1, \chi_4) = (-1)^j \frac{E_{2j}}{2(2j)!} \left(\frac{\pi}{2}\right)^{2j+1},$$

where the E_{2j} are the Euler numbers given by

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n.$$

We have also used the notation

$$\eta(s) = (1 - 2^{1-s}) \zeta(s);$$

and defined

$$\eta_0(s) := (1 - 2^{-s})\zeta(s).$$

The polynomial families $Y_k(z)$ and $S_k(z)$ were studied with the aid of Schinzel's result (9) in [11] and do not need further consideration here, by virtue of the fact that Corollary 5 follows from Theorem 4. We proceed to outline the proofs for the other polynomials Q_k and W_k . It is easy to prove equivalent results to those of Lemma 9 for the other functions. Here we give the corresponding bounds for η and η_0 (the functions involved in $Q_k(z)$ and $W_k(z)$), without proof.

For $Q_k(z)$, we let $q_j = \frac{\eta_0(2j)\eta_0(2k-2j)}{\eta_0(2k-1)}$ for $j = 0, 1, \dots, k$ and $\delta_j = q_j - \eta_0(2j)$. On the other hand, for $W_k(z)$, we put $q_j = \frac{\eta(2j)\eta(2k-2j)}{\eta(2k)}$ for $j = 0, 1, \dots, k$ and $\delta_j = \eta(2j) - q_j$. As before, $\delta_0 = 0$ and $\delta_j > 0$ for $0 < j < k$ for both η and η_0 .

Lemma 11. (i) For $n \geq 2$,

$$1 < \eta_0(n) < 1 + 2^{-n};$$

$$1 - 2^{1-n} < \eta(n) < 1.$$

(ii) For $k \geq 2$ and $j = 1, 2, \dots, k-1$ we have

$$0 < \frac{\eta_0(2k-2j)}{\eta_0(2k-1)} - 1 < 2^{-2k+2j};$$

$$0 < 1 - \frac{\eta(2k-2j)}{\eta(2k)} < 2^{1-2k+2j}.$$

(iii) For $j = 1, 2, \dots, k-1$, we have,

$$0 < \delta_j < 2^{1-2k+2j},$$

for both η_0 and η .

(iv) For $k \geq 2$ and $j = 1, 2, \dots, k-1$ we have $q_j = q_{k-j}$ and

$$|q_{j-1} - q_j| < 2^{2-2j} + 2^{2-2k+2j} \text{ for } \eta_0;$$

$$|q_{j-1} - q_j| < 2^{4-2j} + 2^{2-2k+2j} \text{ for } \eta.$$

(v) For $r \geq 1$, we have,

$$\sum_{j=1}^r |\delta_{j-1} - \delta_j| < \frac{2}{3} \cdot 4^{r+1-k},$$

for both η_0 and η .

(vi)

$$\sum_{j=r+1}^{\lfloor \frac{k}{2} \rfloor} |q_{j-1} - q_j| < \frac{4}{3} \cdot (2^{-2r} + 2^{2-k}) \text{ for } \eta_0;$$

$$\sum_{j=r+1}^{\lfloor \frac{k}{2} \rfloor} |q_{j-1} - q_j| < \frac{16}{3} \cdot (2^{-2r} + 2^{-k}) \text{ for } \eta.$$

4.1. **The polynomial Q_k .** To study Q_k , let us consider the monic polynomial

$$\begin{aligned} N_k(z) &= \frac{1}{2^{2k-1}\eta_0(2k-1)} Q_k(z)(z^2 + 1) \\ &= z^{2k+1} + z^{2k-1} + (-1)^k z^3 + (-1)^k z + \frac{4(-1)^k}{\pi} \sum_{j=1}^k (q_{j-1} - q_j)(-1)^{j-1} z^{2j}. \end{aligned}$$

We can then write a similar decomposition

$$N_k(z)z^{-k-1} = z^{-k}H_r(z) + z^{-k}E_r(z).$$

The main term is given by

$$H_r(z) = z^{2k-2r}h_r(z) + (-1)^k h_r^*(z),$$

where

$$h_r(z) = z^{2r} + z^{2r-2} + \frac{4}{\pi} \sum_{j=1}^r (\eta_0(2j-2) - \eta_0(2j))(-1)^{j-1} z^{2r-2j+1}.$$

On the other hand, the error term is given by

$$E_r(z) = z^k e_r(z) + (-1)^k z e_r^*(z),$$

where

$$e_r(z) = \frac{4}{\pi} \sum_{j=r+1}^{\lfloor \frac{k}{2} \rfloor} (q_{j-1} - q_j)(-1)^{j-1} z^{-2j+k+1} + \frac{4}{\pi} \sum_{j=1}^r (\delta_{j-1} - \delta_j)(-1)^{j-1} z^{-2j+k+1}.$$

From this, for $|z| = 1$,

$$|e_r(z)| \leq \frac{16}{3\pi}(2^{-2r} + 2^{2-k}) + \frac{2}{3\pi} \cdot 4^{r+2-k} \leq 0.14$$

for $r = 2, k \geq 8$.

We need to consider

$$h_2(z) = z^4 - \frac{\pi}{2}z^3 + z^2 + \left(\frac{\pi^3}{24} - \frac{\pi}{2}\right)z.$$

It is not hard to verify that all the zeros have absolute value strictly less than 1, and that, for z on the unit circle, we find that

$$|h_2(z)| \geq |h_2(1)| = \frac{\pi^3}{24} - \pi + 2 = 0.1503\dots$$

Thus Lemma 7 can be applied, using h_2 , e_2 and $c = 0.15$. This finishes the proof for $Q_k(z)$.

4.2. The polynomial W_k . To study W_k , let us consider the monic polynomial

$$\begin{aligned} V_k(z) &= \frac{\pi}{2^{2k}\eta(2k)} W_k(z)(z^2 + 1) \\ &= z^{2k+2} + (-1)^k + 2(-1)^k \sum_{j=1}^k (q_{j-1} - q_j)(-1)^{j-1} z^{2j}. \end{aligned}$$

We can then write

$$V_k(z)z^{-k-1} = z^{-k-1}H_r(z) + z^{-k-1}E_r(z).$$

In this case the main term is given by

$$H_r(z) = z^{2k+2-2r}h_r(z) + (-1)^k h_r^*(z),$$

where

$$h_r(z) = z^{2r} + 2 \sum_{j=1}^r (\eta(2j-2) - \eta(2j))(-1)^{j-1} z^{2r-2j}.$$

The error term is given by

$$E_r(z) = z^{k+1}e_r(z) + (-1)^k z^2 e_r^*(z),$$

where

$$e_r(z) = 2 \sum_{j=r+1}^{\lfloor \frac{k}{2} \rfloor} (q_{j-1} - q_j)(-1)^{j-1} z^{-2j+k+1} - 2 \sum_{j=1}^r (\delta_{j-1} - \delta_j)(-1)^{j-1} z^{-2j+k+1}.$$

From this, for $|z| = 1$,

$$|e_r(z)| \leq \frac{32}{3}(2^{-2r} + 2^{-k}) + \frac{1}{3} \cdot 4^{r+2-k} \leq 0.5$$

for $r = 3, k \geq 6$.

We thus need to consider the polynomial.

$$h_3(z) = z^6 + \left(1 - \frac{\pi^2}{6}\right) z^4 + \left(\frac{7\pi^4}{360} - \frac{\pi^2}{6}\right) z^2 + \left(\frac{7\pi^4}{360} - \frac{31\pi^6}{15120}\right),$$

which has all its zeros in $|z| < 1$. Furthermore, for z on the unit circle, we find that

$$|h_3(z)| \geq |h_3(1)| = |h_3(-1)| = -\frac{31\pi^6}{15120} + \frac{7\pi^4}{180} - \frac{\pi^2}{3} + 2 = 0.5271 \dots$$

So, again, Lemma 7 can be applied, using h_3 , e_3 and $c = 0.52$. This concludes the proof for $W_k(z)$.

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DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE, UNIVERSITÉ DE MONTRÉAL,
CP 6128, SUCC. CENTRE-VILLE, MONTREAL, QC, H3C 3J7, CANADA
E-mail address: mlalin@dms.umontreal.ca

SCHOOL OF MATHEMATICS AND MAXWELL INSTITUTE FOR MATHEMATICAL SCI-
ENCES, UNIVERSITY OF EDINBURGH, EDINBURGH, EH9 3JZ, SCOTLAND
E-mail address: c.smyth@ed.ac.uk